Elastic-Plastic Deformations

Worked Example 2

Department of Mechanical, Materials & Manufacturing Engineering

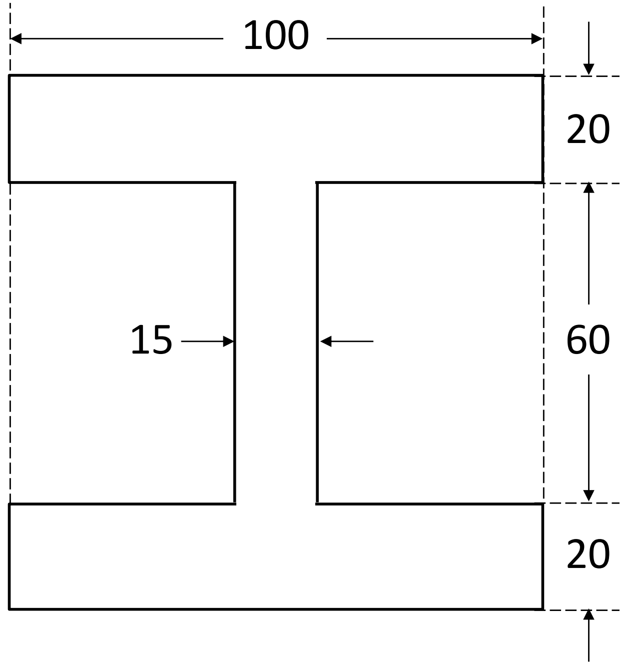
MMME2053 – Mechanics of Solids

University of Nottingham
UK | CHINA | MALAYSIA

Worked Example 2

I-Beam Subjected to a Pure Bending Moment

The problem shown below is the example I-beam subjected to a pure bending moment shown in section 4 of the notes.



All dimensions in mm

Problem

Calculate:

a. the maximum allowable value of M if the web of the section is not to be subjected to any plasticity.

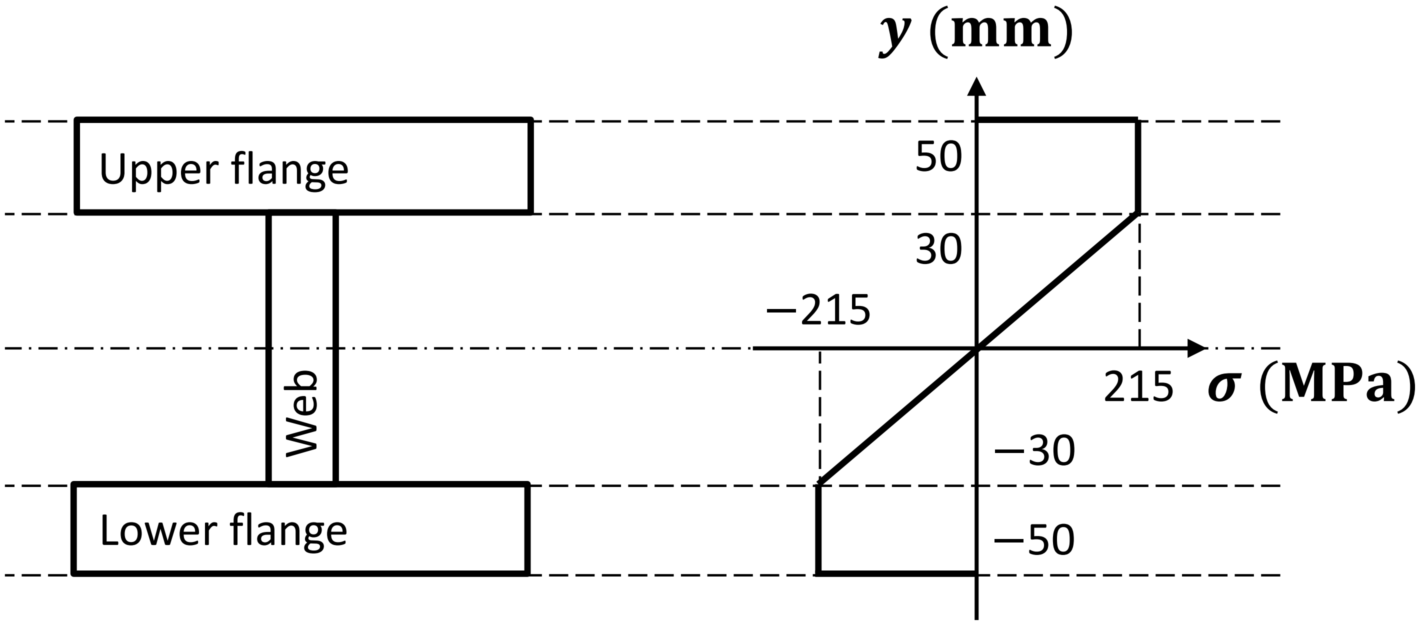
b. the stress distribution and radius of curvature upon the application of M

c. the stress distribution and radius of curvature upon unloading

The material can be assumed to be elastic-perfectly-plastic with a yield stress, $\sigma_{\mathrm{y}} = 215\ \mathrm{MPa}$, and Young's Modulus, $E = 200\ \mathrm{GPa}$.

Stress Distribution

As it is known that the full depths of each flange are allowed to yield, but the web is to remain fully elastic, the resulting stress distribution due to the application of the maximum allowable bending moment, M, is as shown below.



Moment Equilibrium

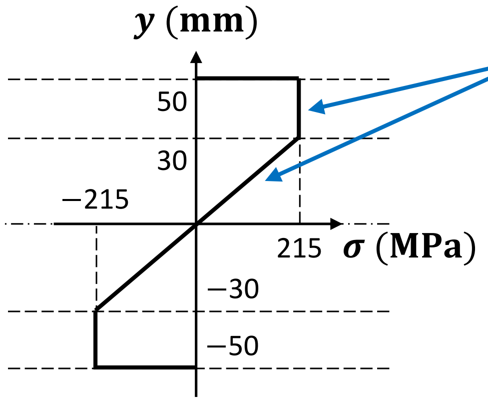
Balancing the moments due to stresses in the elastic and plastic regions with the applied moment:

$$M = \int_A y\sigma \mathrm{d}A = \int_{-d/2}^{d/2} y\sigma b \mathrm{d}y = 2\int_0^{d/2} y\sigma b \mathrm{d}y$$

Substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left(\int_0^a y\left(\sigma_{\mathrm{y}}\frac{y}{a}\right) b_w \mathrm{d}y + \int_a^{d/2} y\sigma_{\mathrm{y}} b_f \mathrm{d}y\right) = 2\sigma_{\mathrm{y}}\left(\frac{b_w}{a}\int_0^a y^2 \mathrm{d}y + b_f\int_a^{d/2} y \mathrm{d}y\right)$$

$$= 2\sigma_{\mathrm{y}}\left(\frac{b_w}{a}\left[\frac{y^3}{3}\right]_0^a + b_f\left[\frac{y^2}{2}\right]_a^{d/2}\right) = 2\sigma_{\mathrm{y}}\left(b_w\frac{a^2}{3} + b_f\left(\frac{d^2}{8} - \frac{a^2}{2}\right)\right)$$



$$\therefore \boldsymbol{M = 36{,}335{,}000\ \mathrm{Nmm} = 36.34\ \mathrm{kNm}}$$

Compatibility

As the region of the cross-section between $-a < y < a$ has only behaved elastically, the elastic beam bending equation can be applied and rearranged to give:

$$R = \frac{y}{\varepsilon}$$

As the beam behaves as one body, this expression for R can be applied to any value of y.

Stress-Strain Relationship

Again, as the region of the cross-section between $-a < y < a$ has only behaved elastically, Hooke's law applies here, which can be substituted into the above expression for R to give:

$$R = \frac{Ey}{\sigma}$$

Substituting values for y and σ from the outermost point of the elastic region gives:

$$R_{\text{load}} = \frac{Ea}{\sigma_y} = 27{,}906.98\ \text{mm} = \mathbf{27.91\ m}$$

Unloading

Assuming that the stress change caused by unloading is purely elastic, then from the elastic beam bending equation:

$$\Delta\sigma = \frac{\Delta M \times y}{I}$$

Therefore, at the top and bottom edges:

$$\Delta\sigma_{(y={}^{d}/_{2})} = \frac{-M \times {}^{d}/_{2}}{I} = \frac{-Md}{2I} = \frac{-36{,}335{,}000 \times 100}{2 \times 6{,}803{,}333.33} = -267.04\ \mathrm{MPa}$$

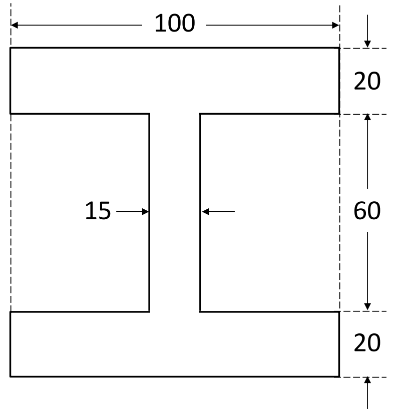
and

$$\Delta\sigma_{(y=-{}^{d}/_{2})} = \frac{-M \times -{}^{d}/_{2}}{I} = \frac{Md}{2I} = 267.04\ \mathrm{MPa}$$

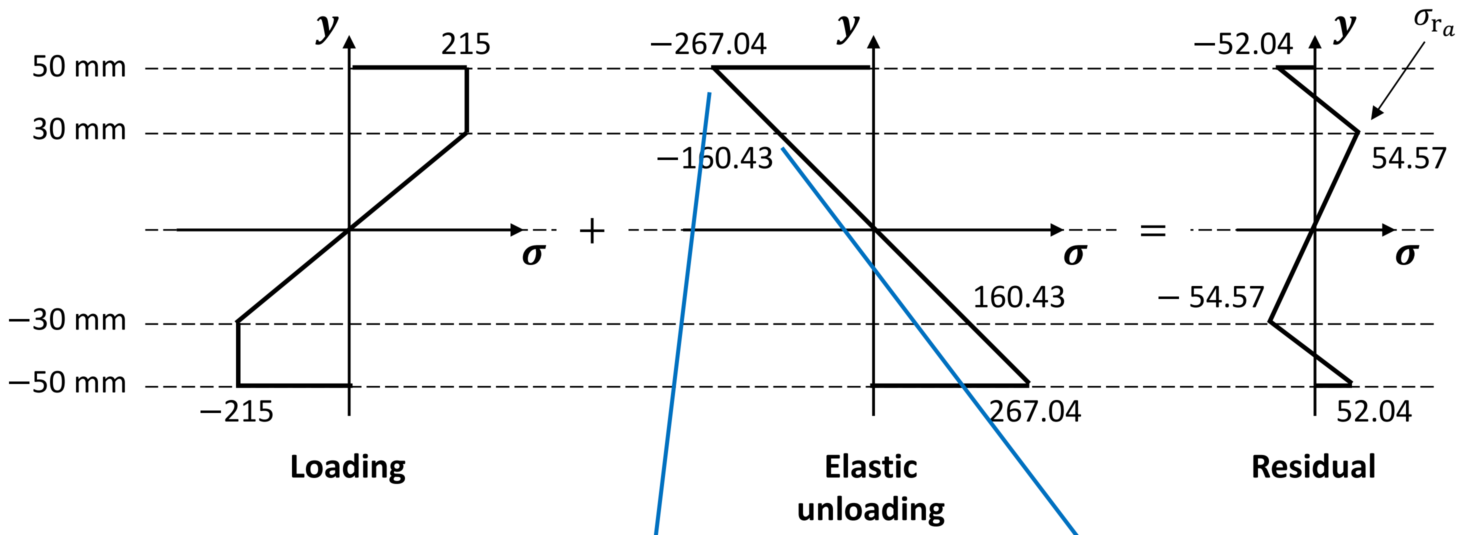
Where:

$$I = \left(\frac{bd^3}{12}\right)_{outer} - \left(\frac{bd^3}{12}\right)_{gaps} = \frac{100 \times 100^3}{12} - 2\left(\frac{42.5 \times 60^3}{12}\right) = 8{,}333{,}333.33\ \mathrm{mm}^4 - 1{,}530{,}000\ \mathrm{mm}^4$$

$$= 6{,}803{,}333.33\ \mathrm{mm}^4$$



Since the unloading behaviour has been assumed to be elastic, the variation between the stress change values, relating to the top and bottom edges, will be linear, as shown in the unloading section of the figure below.



The loading and unloading stress distributions can now be summed together in order to determine the residual stress distribution.

The unloading curve can be interpolated to find unloading stress change values at $y = \pm 30$ mm as follows:

$$y = m\sigma + \cancelto{0}{c}$$

$$m = \frac{y}{\sigma} = \frac{50}{-267.04} = -0.187$$

$$\sigma = \frac{30}{-0.187} = -160.43$$

This residual stress distribution will be accompanied by a residual radius of curvature, which can be calculated by substituting unloaded beam values for y and σ into the expression derived for R, which again relate to a position that has only been subjected to elastic behaviour.

As before (under loaded conditions), a convenient value of y to use is $a = 30\ \mathrm{mm}$ (which is the outermost point of the elastic region), for which the corresponding value of σ can be taken from the residual stress distribution to be $54.57\ \mathrm{MPa}$.

I.e.,

$$R = \frac{Ey}{\sigma}$$

$$\therefore R_{\mathrm{unload}} = \frac{Ea}{\sigma_{\mathrm{r}_a}} = 109{,}950.52\ \mathrm{mm} = \mathbf{109.95\ m}$$